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Reg. No................................

Name....................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2015

Third Semester

Faculty of Science

Branch I-(A) : Mathematics

MT 03 C13—DIFFERENTIAL GEOMETRY

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Show that the graph of any function $f : \mathrm{R}^n \to \mathrm{R}$ is a level set for some function $\mathrm{F} : \mathrm{R}^{n+1} \to \mathrm{R}$.

2. Show that the gradient of f at $\mathrm{P} \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at P.

3. Define a geodesic, show that the parametrized curve $\alpha(t) = (\cos(at+b), \sin(at+b), (ct+d)$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in R^3.

4. Define Gauss Map and Spherical image of an oriented n-surface S.

5. Show that $\nabla_{\mathrm{v}}(\mathrm{X}+\mathrm{Y}) = \nabla_{\mathrm{v}}\mathrm{X} + \nabla_{\mathrm{v}}\mathrm{Y}$ for all smooth vector fields X and Y on S.

6. Define a global parametrization of an oriented plane curve C.

7. Find the normal curvature $k(v)$ for each tangent direction v at the point $p = (1, 0 \ldots\ldots\ldots 0)$ of $x_1 + x_2 \ldots\ldots\ldots + x_{n+1} = 1$ oriented by $\nabla f / \|\nabla f\|$, where

$$f(x_1, x_2 \ldots\ldots\ldots x_{n+1}) = x_1 + x_2 + \ldots\ldots\ldots + x_{n+1}.$$

8. Define a parametrized n-surface in R^{n+k}. Give an example.

(5 × 1 = 5)

Turn over

Part B

*Answer any **five** questions.*
Each question has weight 2.

9. Find and sketch the gradient field of $f(x_1, x_2) = x_1 - x_2^2$.

10. Show that the cylinder $x_1^2 + x_2^2 = 1$ can be represented as a level set of $f(x_1, x_2, x_3) = -x_1^2 - x_2^2$.

11. Describe the spherical image when $n = 1$ and when $n = 2$ of the n surface $x_2^2 + \dots\dots\dots x_{n+1}^2 = 1$ oriented by $\nabla f \| \nabla f \|$ where f is the function $x_2^2 + \dots\dots\dots + x_{n+1}^2$.

12. Find the velocity, the acceleration and the speed of $\alpha(t) = (\cos t, \sin t, t)$.

13. Compute $\nabla_{\mathrm{v}} \mathrm{X}$ where $\mathrm{V} \in \mathrm{R}_p^{n+1}$, $\mathrm{X}(x_1, x_2) = \left(x_1, x_2, x_1 x_2, x_2^2\right)$, $\mathrm{V} = (1, 0, 0, 1)$ and $n = 1$.

14. For each 1-form w on $\mathrm{U}\left(\mathrm{U} \text{ open in } \mathrm{R}^{n+1}\right)$ show that there exist unique functions $f_i : \mathrm{U} \to \mathrm{R}\ (i \in \{1, 2, \dots\dots n+1\})$ such that $w = \sum\limits_{i=1}^{n+1} f_i \, d x_i$.

15. Find the Gaussian curvature $\mathrm{K} : \mathrm{S} \to \mathrm{R}$ where S is the surface given by $x_1^2 + x_2^2 - x_3^2 = 0$, $x_3 > 0$.

16. State and prove inverse function theorem for n-surfaces.

$(5 \times 2 = 10)$

Part C

*Answer any **three** questions.*
Each question has weight 5.

17. Let U be an open set in R^{n+1} and let $f : \mathrm{U} \to \mathrm{R}$ be smooth. Let $p \in \mathrm{U}$ be a regular point of f and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

18. Let S be an n-surface in R^{n+1}, let $p \in \mathrm{S}$, and let $v \in \mathrm{S}_p$ Prove that there exists an open interval I containing o and a geodesic $\alpha : \mathrm{I} \to \mathrm{S}$ such that (i) $\alpha(0) = p$ and $\dot{\alpha}(0) = v$; (ii) If $\beta : \bar{\mathrm{I}} \to \mathrm{S}$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\bar{\mathrm{I}} \subset \mathrm{I}$ and $\beta(t) = \alpha(t)$ for all $t \in \bar{\mathrm{I}}$.

19. Prove that the Weingarten map L_p is self-adjoint.

20. Let η be the 1-form on $R^2 - \{0\}$ defined by $\eta = -\frac{x^2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$.

Prove that for $\alpha : \{a, b\} \to R^2 - \{0\}$, any closed piecewise smooth parametrized curve in $R^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$ for some integer k.

21. Let $\phi = U \to R^{n+1}$ be a parametrized n-surface in R^{n+1} and let $p \in U$.

Prove that there exists an open set $U_1 \subset U$ about P such that $\phi(U_1)$ is an n-surface in R^{n+1}.

22. Prove that on each compact oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite.

$(3 \times 5 = 15)$