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M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2015

Third Semester

Faculty of Science

Branch I (A)—Mathematics

MT 03 C12—FUNCTIONAL ANALYSIS

(2012 Admission Onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. If a normed space X has the property that the closed unit ball is compact. Show that X is finite dimensional.

2. Show that if a normed space X is finite dimensional, then every linear operator on X is bounded.

3. Show that a finite dimensional vector space is algebraically reflexive.

4. Show that in an inner product space $x \perp y$ if and only if $\|x + \alpha\, y\| \geq \|x\|$ for all scalars α.

5. Let y be any closed subspace of a Hilbert space H. Show that $H = Y \oplus Y^{\perp}$.

6. Let A and $B \supset A$ be non-empty subsets of an inner product space X.

   Show that : (a) $A \subset A^{\perp\perp}$. (b) $B^{\perp} \subset A^{\perp}$.

7. Pove that for every x in a normed space X, $\|x\| = \sup\limits_{\substack{f \in X^1 \\ f \neq 0}} \dfrac{|f(x)|}{\|f\|}$.

8. Show that the canonical mapping C defined by $C(x) = g_x$, where $x \in X$ and $gx \in X''$ is an isomorphism of the normed space X onto the normed space $\mathfrak{R}$ (C), the range of C.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Show that every finite dimensional subspace Y of a normed space X is complete.

10. State and prove Riesz's lemma.

Turn over

11. Let $T : \mathscr{D}(T) \to Y$ be a linear operator, where $\mathscr{D}(T) \subset X$ and X and Y are normed spaces. Prove that
    (a) T is continuous if and only if T is bounded.
    (b) If T is continuous at a single point, it is continuous.

12. If Y is Banach space, show that $B(X, Y)$ is a Banach space.

13. Let X be an inner product space and $M \neq \phi$ a convex subset which is complete. Prove that for every $x \in X$ there exists a unique $y \in M$ such that $\delta = \inf_{\bar{y} \in M} \|\bar{y} - x\| = \|y - x\|$.

14. Prove that every Hilbert space H is reflexive.

15. Show that in every Hilbert space $H \neq [0]$ there exists a total orthonormal set.

16. Let X be a normed space and $x_0 \neq 0$ be any element of X. prove that there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1$, $\tilde{f}(x_0) = \|x_0\|$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. Let X be a normed space, show that there is a Banach space $\tilde{X}$ and an isometry A from X onto a subspace W of $\tilde{X}$ which is dense in $\tilde{X}$. Also show that $\tilde{X}$ is unique, except for isometries.

18. Show that the dual of l^p is l^q for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

19. (a) Let X be an n-dimensional vector space and $E = \{e_1, e_2, \ldots e_n\}$ a basis for X. Show that $F = \{f_1, f_2, \ldots f_n\}$ given by $f_k(c_j) = \delta_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$ is a basis for the algebraic dual of X.
    (b) Show that the dual of l^1 is l^∞.

20. State and prove Bessel inequality.

21. Prove that two Hilbert spaces H and $\tilde{H}$, both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.

22. State and prove Generalized Hahn-Banach theorem.

(3 × 5 = 15)